Sémantique des Langages de Programmation (SemLP)
TD n° 6 : Simulation

Exercice 1 :

Prove the following properties :

1. $\leqslant_C$ is a pre-order (reflexive and transitive).

2. If $M \leqslant_C N$ then for all contexts C (not necessarily closing) $C[M] \leqslant_C C[N]$.

3. $\lambda x.\lambda y.x \nleqslant_C \lambda x.\lambda y.y$.

4. Find a pair of λ-terms M, N such that $M \approx_C N$ and $M \neq_\beta N$.

Exercice 2 :

Let $\leqslant_{IO}$ be a relation on closed λ-terms defined by :

$$M \leqslant_{IO} N \text{ if } \forall P \text{ closed } MP \Downarrow \text{ implies } NP \Downarrow$$

Show that $\leqslant_{IO}$ is a pre-order and that it is *not* preserved by contexts.

Exercice 3 :

Show that :

1. The *subsets of a set* with the inclusion relation as partial order form a complete lattice.

2. Every subset of a complete lattice has an *inf.*

3. Every *finite* subset of a lattice has an *inf.*

4. Every *finite lattice* is *complete.*

Exercice 4 :

Let $(\mathbb{N} \cup \{\infty\}, \leqslant)$ be the set of natural numbers with an added maximum element ∞, $0 < 1 < 2 < \ldots < \infty$. Show that every monotonic function f on this order has a fixed point.

Exercice 5 :

Let $(L, \leqslant)$ be a *finite* lattice and $f : L \to L$ be a monotonic function. Let $\bot$ ($\top$) be the *least* (*greatest*) element of L. If $x \in L$ then let $f^n(x)$ be the *n-time iteration* of f on x, where $f^0(x) = x$.

1. Show that there is an $n \geqslant 0$ such that the *least fixed point* of f equals $f^n(\bot)$.

2. State and prove a dual property for the *greatest fixed point.*

3. Show that these properties fail to hold if one removes the hypothesis that the lattice is *finite.*

Exercice 6 :
A subset X of a partial order is *directed* if

$$\forall\, x, y \in X, \exists\, z \in X,\ (x \leqslant z) \text{ and } (y \leqslant z)\ .$$

A function on a complete lattice is *continuous* if it preserves the *sup* of *directed sets* :

$$f(sup(X)) = sup(f(X)) \qquad \text{(if } X \text{ directed).}$$

1. Show that a *continuous* function is *monotonic*.
2. Give an example of a function on a complete lattice which is continuous but does *not* preserve the *sup* of a (non-directed) set.
3. Show that the *least fixed point* of a continuous function f is expressed by :

$$sup\{f^n(\bot) \mid n \geqslant 0\}\ .$$

Exercice 7 :
Prove that :
1. If $M \not\Downarrow$ and $N \not\Downarrow$ then $M =_S N$.
2. Let $\Omega_n = \lambda x_1. \ldots .\lambda x_n.\Omega$. Then $\Omega_n <_S \Omega_{n+1}$ (strictly) and, for all M, $\Omega_0 \leqslant_S M$.
3. Let $K^\infty \equiv YK$. Then for all M, $M \leqslant_S K^\infty$.
4. $\lambda x.x \not\leqslant_S \lambda x, y.xy$ (thus η-conversion is unsound).

Exercice 8 :
Let us revise the pre-order considered in exercise 2 by defining a relation $\leqslant_{IO^*}$ on closed λ-terms as :

$$M \leqslant_{IO^*} N \text{ if for all } n \geqslant 0,\ P_1, \ldots, P_n \text{ closed, } MP_1 \cdots P_n \Downarrow \text{ implies } NP_1 \cdots P_n \Downarrow.$$

Prove that $\leqslant_{IO^*}$ coincides with $\leqslant_S$.